

THE TOPOLOGY ON BERKOVICH AFFINE LINES OVER COMPLETE VALUATION RINGS

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ABSTRACT. In this article, we give a full description of the topology of the one dimensional affine analytic space \mathbb{A}_R^1 over a complete valuation ring R (i.e. a valuation ring with “real valued valuation” which is complete under the induced metric), when its field of fractions K is algebraically closed. In particular, we show that \mathbb{A}_R^1 is both connected and locally path connected. Furthermore, \mathbb{A}_R^1 is the completion of $K \times (1, \infty)$ under a canonical uniform structure. As an application, we describe the Berkovich spectrum $\mathcal{M}(\mathbb{Z}_p[G])$ of the Banach group ring $\mathbb{Z}_p[G]$ of a cyclic p -group G over the ring \mathbb{Z}_p of p -adic integers.

1. INTRODUCTION AND NOTATION

Let S be a commutative unital Banach ring with its norm being denoted by $\|\cdot\|$. The *Berkovich spectrum* $\mathcal{M}(S)$ as well as the *n -dimensional affine analytic space* \mathbb{A}_S^n over S was introduced by Vladimir Berkovich (see, e.g., [3]) and was further studied by Jérôme Poineau in [8]. More precisely, $\mathcal{M}(S)$ is the set of all non-zero contractive multiplicative semi-norms on S , while \mathbb{A}_S^n is the set of all non-zero multiplicative seminorms on the n -variables polynomial ring $S[\mathbf{t}_1, \dots, \mathbf{t}_n]$ whose restrictions on S are contractive. The topologies on both $\mathcal{M}(S)$ and \mathbb{A}_S^n are the ones given by pointwise convergence. The topology on \mathbb{A}_S^n is also induced by the *Berkovich uniform structure* which is given by a fundamental system of entourages consisting of sets of the form

$$E_\epsilon^X := \{(\mu, \nu) \in \mathbb{A}_S^n \times \mathbb{A}_S^n : |\mathbf{p}|_\mu - |\mathbf{p}|_\nu| < \epsilon, \text{ for any } \mathbf{p} \in X\}, \quad (1)$$

where ϵ runs through all strictly positive real numbers and X runs through all non-empty finite subsets of $S[\mathbf{t}_1, \dots, \mathbf{t}_n]$. It is not hard to see that \mathbb{A}_S^n is complete under this uniform structure.

In the case of a non-Archimedean field L , the properties of \mathbb{A}_L^n plays an important role in the study of non-Archimedean geometry. For example, the Bruhat-Tits tree of $SL_2(\mathbb{Q}_p)$ can be realized as a subspace of the Berkovich projective line, which is a glue of two copies of $\mathbb{A}_{\mathbb{Q}_p}^1$ (see [10]).

When L is an algebraically closed non-Archimedean complete valued field, Berkovich gave in [3] a full description of the space \mathbb{A}_L^1 . This may then be used to describe the one-dimensional affine analytic spaces over not necessarily algebraically closed fields.

The aim of this article is to give a full description of the topology space \mathbb{A}_R^1 of a “complete valuation ring” R . Recall that an integral domain R is a *valuation ring* if for every element x in its field of fractions K , either $x \in R$ or $x^{-1} \in R$ (see e.g. [1, p.65] or Definition 2 of [4, §VI.1]). It is easy to see that if R^\times and K^\times are the sets of invertible elements in R and K respectively, then K^\times/R^\times is a totally ordered abelian group and the canonical map $\nu_R : K^\times \rightarrow K^\times/R^\times$ is a “valuation” such that $R = \{x \in K^\times : \nu_R(x) \geq 0\} \cup \{0_R\}$.

Definition 1.1. A valuation ring R is called a complete valuation ring if K^\times/R^\times is isomorphic to an ordered subgroup of \mathbb{R} and R is complete under the induced norm.

Date: March 17, 2017.

2010 Mathematics Subject Classification. Primary: 32P05, 32C18, 13F30; Secondary: 12J25, 13A18, 13F20.

Key words and phrases. complete valued fields, valuation rings, affine analytic spaces, Berkovich spectrum, Banach group rings.

If R is a complete valuation ring, then K is a non-Archimedean complete valued field and R coincides with the ring of integers, $\{a \in K : |a| \leq 1\}$, of K . Conversely, the ring of integers of a non-Archimedean complete valued field is always a complete valuation ring.

Throughout this article, for a commutative unital Banach ring S , we denote

$$S^* := S \setminus \{0_S\},$$

where 0_S is the zero element of S . The identity of S will be denoted by 1_S . For any $n \in \mathbb{N}$, we define

$$S\{n^{-1}\mathbf{t}\} := \left\{ \sum_{k=0}^{\infty} a_k \mathbf{t}^k : \lim_k \|a_k\| n^k = 0 \right\}$$

and equip it with the norm $\|\sum_{k=0}^{\infty} a_k \mathbf{t}^k\| := \max_k \|a_k\| n^k$. It is well-known that $S\{n^{-1}\mathbf{t}\}$ is a commutative unital Banach ring. For simplicity, we will use the notation $S\{\mathbf{t}\}$ for $S\{1^{-1}\mathbf{t}\}$.

It was shown in [3] that for each $n \in \mathbb{N}$, the compact space $\mathcal{M}(S\{n^{-1}\mathbf{t}\})$ can be identified with the subspace $\{\mu \in \mathbb{A}_S^1 : |\mathbf{t}|_{\mu} \leq n\}$ of \mathbb{A}_S^1 . If we set

$$\mathbb{U}_n^S := \{\mu \in \mathbb{A}_S^1 : |\mathbf{t}|_{\mu} < n\}, \quad (2)$$

then $\mathbb{A}_S^1 = \bigcup_{n \in \mathbb{N}} \mathbb{U}_n^S = \bigcup_{n \in \mathbb{N}} \mathcal{M}(S\{n^{-1}\mathbf{t}\})$ and hence \mathbb{A}_S^1 is both locally compact and σ -compact.

From now on, R is a complete valuation ring and K is its field of fractions. The absolute value on K induced by ν_R will be denoted by $|\cdot|$. The residue field of R is denoted by F and $Q : R \rightarrow F$ is the quotient map. We set

$$\tilde{Q} : R[\mathbf{t}] \rightarrow F[\mathbf{t}] \quad (3)$$

to be the map induced by Q .

Suppose that $s \in K$ and $\mathbf{p} \in K[\mathbf{t}]$. If $r_0, \dots, r_n \in K$ are the unique elements with $\mathbf{p} = \sum_{k=0}^n r_k \mathbf{t}^k$, then we put $\mathbf{p}_s := \sum_{k=0}^n r_k (\mathbf{t} + s)^k$. For any $\lambda \in \mathbb{A}_K^1$, we define

$$|\mathbf{p}|_{\lambda+s} := |\mathbf{p}_s|_{\lambda} \quad (\mathbf{p} \in K[\mathbf{t}]) \quad \text{and} \quad \lambda - s := \lambda + (-s). \quad (4)$$

It is easy to see that $\lambda + s \in \mathbb{A}_K^1$, and that $\lambda \mapsto \lambda + s$ is a bicontinuous bijection from \mathbb{A}_K^1 to itself.

For every $s \in K$ and $\tau \in \mathbb{R}_+$, we denote the closed ball with center s and radius τ by $D(s, \tau)$, i.e.

$$D(s, \tau) := \{t \in K : |t - s| \leq \tau\},$$

and define (as in [2]) $\zeta_{s,\tau} \in \mathbb{A}_K^1$ by $|\mathbf{p}|_{\zeta_{s,\tau}} := \sup_{t \in D(s,\tau)} |\mathbf{p}(t)|$ ($\mathbf{p} \in K[\mathbf{t}]$). Because of the maximum modulus principle, one has

$$\left| \sum_{k=0}^n a_k (\mathbf{t} - s)^k \right|_{\zeta_{s,\tau}} = \max_{k=0,\dots,n} |a_k| \tau^k \quad (n \in \mathbb{N}; a_0, \dots, a_n \in K); \quad (5)$$

here, we use the convention that $0^0 := 1$.

Let us recall the following result from [3].

Theorem 1.2. (*Berkovich*) *Let L be an algebraically closed non-Archimedean complete valued field with a non-trivial norm $|\cdot|$, and $|\cdot|_{\lambda} : L[\mathbf{t}] \rightarrow \mathbb{R}_+$ be a function. Then $\lambda \in \mathbb{A}_L^1$ if and only if there is a decreasing sequence $\{D(s_n, \tau_n)\}_{n \in \mathbb{N}}$ of closed balls in L such that $|\mathbf{p}|_{\lambda} = \inf_{n \in \mathbb{N}} |\mathbf{p}|_{\zeta_{s_n, \tau_n}}$.*

The topology on \mathbb{A}_L^1 was also described in [3]. Moreover, as noted in [3], by using the fact that $D(t, \tau) = D(s, \tau)$ whenever $D(s, \tau) \subseteq D(t, \tau)$, one can show easily that \mathbb{A}_L^1 is path connected.

It is also well-known that \mathbb{A}_L^1 is *locally path connected*, in the sense that for every point in this topological space, there is a local neighborhood basis at that point consisting of open sets that are path connected under the induced topologies. Indeed, as noted in [2], any two points in \mathbb{A}_L^1 are joined by a unique path, and hence \mathbb{A}_L^1 is a \mathbb{R} -tree. The “weak topology” induced by this \mathbb{R} -tree structure coincides with the pointwise convergence topology on $\mathcal{M}(L\{\mathbf{t}\})$ (see e.g. [2, Proposition 1.13]). Since the canonical basic neighborhoods of the “weak topology” are path connected, we know

that $\mathbb{U}_1^L \subseteq \mathcal{M}(L\{\mathfrak{t}\})$ (see (2)) is locally path connected. Furthermore, as $\zeta_{s,0} = \zeta_{0s,0} + s \in \mathbb{U}_1^L + s$ (see (4)), the density of the image of L in \mathbb{A}_L^1 implies $\mathbb{A}_L^1 = \bigcup_{s \in L} \mathbb{U}_1^L + s$, and this gives the local path connectedness of \mathbb{A}_L^1 .

Observe that if the complete valuation ring R is actually a field, then the absolute value $|\cdot|$ on K is trivial, and the structure of \mathbb{A}_R^1 is already given in [3, 1.4.4]. However, because we need a concrete presentation of this space for the general case, we will first have a closer look at this case in Proposition 2.1. As a sidetrack, we verify the fact that if two fields \mathfrak{k}_1 and \mathfrak{k}_2 are endowed with the trivial norm, then $\mathbb{A}_{\mathfrak{k}_1}^1 \cong \mathbb{A}_{\mathfrak{k}_2}^1$ if and only if the cardinalities of the sets of monic irreducible polynomials over them are the same.

Suppose that R is not a field, or equivalently, the absolute value $|\cdot|$ on K is non-trivial. Let us pick an arbitrary number $\omega \in [1, \infty)$, and

set K^ω to be the field K equipped with the equivalent norm $|\cdot|^\omega$.

As $|a|^\omega \leq |a|$ ($a \in R$), we know that every semi-norm $\lambda \in \mathbb{A}_{K^\omega}^1$ restricts to an element in \mathbb{A}_R^1 and this gives a map

$$J_\omega^\mathbb{A} : \mathbb{A}_{K^\omega}^1 \rightarrow \mathbb{A}_R^1.$$

The map $J_\omega^\mathbb{A}$ is injective, because for any $\mathfrak{p} \in K[\mathfrak{t}]$, there exists $a \in R$ with $a\mathfrak{p} \in R[\mathfrak{t}]$. On the other hand, the surjection \hat{Q} as in (3) produces an injection

$$Q^\mathbb{A} : \mathbb{A}_F^1 \rightarrow \mathbb{A}_R^1.$$

It is not hard to see that one actually has $\mathbb{A}_R^1 = Q^\mathbb{A}(\mathbb{A}_F^1) \cup \bigcup_{\omega \in [1, \infty)} J_\omega^\mathbb{A}(\mathbb{A}_{K^\omega}^1)$ (Proposition 2.4). Note, however, that in the case of a general Banach integral domain S , elements in \mathbb{A}_S^1 cannot be described in such an easy way; for example, if S is the ring \mathbb{Z} equipped with the trivial norm, the description of elements in \mathbb{A}_S^1 requires the knowledge of all multiplicative ultrametric norms on \mathbb{Q} , instead of just the trivial norm on \mathbb{Q} (which is the one induced from S).

The topology on \mathbb{A}_R^1 is more difficult to describe, and we will give a full presentation of it in Theorem 2.6, in the case when K is algebraically closed. Using this description, we obtain in Theorem 2.8(c) that \mathbb{A}_R^1 is first countable if and only if F is countable and \mathbb{A}_K^1 is first countable. We will also verify, in Proposition 2.10, that \mathbb{A}_R^1 is second countable if and only if R is separable as a metric space (or equivalently, K is a separable metric space). Moreover, the Berkovich uniform space \mathbb{A}_R^1 is the completion of $K \times (1, \infty)$ under the induced uniform structure (see Remark 2.7(c)).

We also show that \mathbb{A}_R^1 is both connected and locally path connected (parts (a) and (b) of Theorem 2.8). Notice that, unlike the case of \mathbb{A}_K^1 , any two points in a connected open subset of \mathbb{A}_R^1 are joined by infinitely many paths inside that subset (see Remark 2.9). Consequently, the topology on \mathbb{A}_R^1 cannot be described using the “weak topology” of a \mathbb{R} -tree structure.

Finally, we will apply our main result to give a description of the Berkovich spectrum of the Banach group ring $R[G]$ of a cyclic group G over R (Corollary 2.13). In the case when K is not necessarily algebraically closed, one may obtain information about $\mathcal{M}(R[G])$ by looking at the corresponding spectrum over the completion of the algebraic closure of K . In particular, we will take a closer look at the case when $R = \mathbb{Z}_p$ and G is a cyclic p -group, for a fixed prime number p (Example 2.15).

2. THE MAIN RESULTS

Let us begin with a careful presentation of the content of the second line of [3, 1.4.4]. More precisely, we will give a concrete description of $\mathbb{A}_{\mathfrak{k}}^1$ when \mathfrak{k} is a field (not necessarily algebraically closed) equipped with the trivial norm.

In the following, $\mathfrak{k}[\mathfrak{t}]_{\text{irr}}$ is the set of all monic irreducible polynomials in $\mathfrak{k}[\mathfrak{t}]$. Consider $\mathbf{q}, \mathbf{q}' \in \mathfrak{k}[\mathfrak{t}]_{\text{irr}}$ as well as $\kappa, \kappa' \in \mathbb{R}_+$. We define a semi-norm $\gamma_{\mathbf{q}, \kappa}$ on $\mathfrak{k}[\mathfrak{t}]$ by

$$\left| \sum_{l=0}^n \mathbf{r}_l \mathbf{q}^l \right|_{\gamma_{\mathbf{q}, \kappa}} := \max_{\mathbf{r}_l \neq 0} \kappa^l \quad (6)$$

(again, $0^0 := 1$), where $\mathbf{r}_0, \dots, \mathbf{r}_{n-1} \in \mathfrak{k}[\mathbf{t}]$ and $\mathbf{r}_n \in \mathfrak{k}[\mathbf{t}]^*$ are elements with degrees strictly less than $\deg \mathbf{q}$. Note that, because the absolute value on \mathfrak{k} is trivial, one has (see (5))

$$\gamma_{\mathbf{t}-x, \kappa} = \zeta_{x, \kappa} \quad (x \in \mathfrak{k}; \kappa \in \mathbb{R}_+). \quad (7)$$

The semi-norm $\gamma_{\mathbf{q}, 1}$ is independent of \mathbf{q} and equals the trivial norm on $\mathfrak{k}[\mathbf{t}]$. Furthermore, when $\gamma_{\mathbf{q}, \kappa} = \gamma_{\mathbf{q}', \kappa'}$, we have $\kappa = \kappa'$, and we will also have $\mathbf{q} = \mathbf{q}'$ if, in addition, $\kappa = \kappa' < 1$. For $\kappa \in (1, \infty)$ and $x \in \mathfrak{k}$, one has $\gamma_{\mathbf{t}-x, \kappa}(\mathbf{p}) = \gamma_{\mathbf{t}, \kappa}(\mathbf{p}) = \kappa^{\deg \mathbf{p}}$ ($\mathbf{p} \in \mathfrak{k}[\mathbf{t}]$).

Notice that $\gamma_{\mathbf{t}, \kappa} \in \mathbb{A}_{\mathfrak{k}}^1$ for any $\kappa \in \mathbb{R}_+$, but $\gamma_{\mathbf{q}, \kappa}$ is not submultiplicative when $\deg \mathbf{q} > 1$ and $\kappa > 1$. Nevertheless, if $\kappa \in [0, 1)$, then $\gamma_{\mathbf{q}, \kappa} \in \mathbb{A}_{\mathfrak{k}}^1$ (regardless of the degree of \mathbf{q}).

On the other hand, for any $\tau \in [-1, \infty)$ and $\mathbf{q} \in \mathfrak{k}[\mathbf{t}]_{\text{irr}}$, we consider $\delta_{\mathbf{q}, \tau}$ to be the function from $\mathfrak{k}[\mathbf{t}]_{\text{irr}}$ to $[-1, \infty)$ that vanishes outside the point \mathbf{q} and sends \mathbf{q} to τ . Observe that $\delta_{\mathbf{q}, 0}$ is the constant zero function for all $\mathbf{q} \in \mathfrak{k}[\mathbf{t}]_{\text{irr}}$.

Proposition 2.1. *Let \mathfrak{k} be a field endowed with the trivial norm. Then $\mathbb{A}_{\mathfrak{k}}^1$ is canonically homeomorphic to the subspace $X := \{\delta_{\mathbf{q}, \tau} : \mathbf{q} \in \mathfrak{k}[\mathbf{t}]_{\text{irr}} \setminus \{\mathbf{t}\}; \tau \in [-1, 0)\} \cup \{\delta_{\mathbf{t}, \tau} : \tau \in [-1, \infty)\}$ of the product space $\prod_{\mathfrak{k}[\mathbf{t}]_{\text{irr}}} [-1, \infty)$. Consequently, $\mathbb{A}_{\mathfrak{k}}^1$ is connected and locally path connected. Moreover, $\mathbb{A}_{\mathfrak{k}}^1$ is first countable if and only if \mathfrak{k} is at most countable.*

Proof. Since the second and the third statements follow easily from the first one, we will only establish the first statement (observe that $\mathfrak{k}[\mathbf{t}]_{\text{irr}}$ is countable if and only if \mathfrak{k} is at most countable). Let us show that

$$\mathbb{A}_{\mathfrak{k}}^1 = \{\gamma_{\mathbf{q}, \kappa} : \mathbf{q} \in \mathfrak{k}[\mathbf{t}]_{\text{irr}} \setminus \{\mathbf{t}\}; \kappa \in [0, 1)\} \cup \{\gamma_{\mathbf{t}, \tau} : \tau \in \mathbb{R}_+\}. \quad (8)$$

In fact, consider any $\lambda \in \mathbb{A}_{\mathfrak{k}}^1$ and set $\tau := |\mathbf{t}|_{\lambda}$. If $\tau \in \mathbb{R}_+ \setminus \{1\}$, one may deduce from the Isosceles Triangle Principle that $\lambda = \gamma_{\mathbf{t}, \tau}$.

Suppose that $\tau = 1$. Clearly, $|\mathbf{p}|_{\lambda} \leq |\mathbf{p}|_{\gamma_{\mathbf{t}, 1}}$ ($\mathbf{p} \in \mathfrak{k}[\mathbf{t}]$) and we consider the case when $\lambda \neq \gamma_{\mathbf{t}, 1}$. Let

$$P^{\lambda} := \{\mathbf{p} \in \mathfrak{k}[\mathbf{t}] : |\mathbf{p}|_{\lambda} < 1\}.$$

As P^{λ} is a non-zero prime ideal of $\mathfrak{k}[\mathbf{t}]$, we know that $P^{\lambda} = \mathbf{q} \cdot \mathfrak{k}[\mathbf{t}]$ for a unique element $\mathbf{q} \in \mathfrak{k}[\mathbf{t}]_{\text{irr}} \setminus \{\mathbf{t}\}$ (note that $|\mathbf{t}|_{\lambda} = \tau = 1$), and we put $\kappa := |\mathbf{q}|_{\lambda} \in [0, 1)$. For any $n \in \mathbb{Z}_+$ and $\mathbf{r}_0, \dots, \mathbf{r}_n \in \mathfrak{k}[\mathbf{t}]$ with $\mathbf{r}_n \neq 0$ and $\deg \mathbf{r}_k < \deg \mathbf{q}$ ($k = 0, \dots, n$), we know from the Isosceles Triangle Principle that $|\sum_{k=0}^n \mathbf{r}_k \mathbf{q}^k|_{\lambda} = |\sum_{k=0}^n \mathbf{r}_k \mathbf{q}^k|_{\gamma_{\mathbf{q}, \kappa}}$.

Next, we define a map $\Phi : \mathbb{A}_{\mathfrak{k}}^1 \rightarrow X$ by $\Phi(\gamma_{\mathbf{q}, \kappa}) := \delta_{\mathbf{q}, \kappa-1}$ ($\gamma_{\mathbf{q}, \kappa} \in \mathbb{A}_{\mathfrak{k}}^1$). Clearly, Φ is bijective, and is continuous on the two subsets $\{\gamma_{\mathbf{q}, \kappa} : \mathbf{q} \in \mathfrak{k}[\mathbf{t}]_{\text{irr}}; \kappa \in [0, 1)\}$ and $\{\gamma_{\mathbf{t}, \tau} : \tau \in [1, \infty)\}$. As Φ restricts to a homeomorphism on the compact subset $\mathcal{M}(\mathfrak{k}[\mathbf{t}])$ of $\mathbb{A}_{\mathfrak{k}}^1$, it is not hard to verify that Φ is actually bicontinuous. \square

If \mathfrak{k} is algebraically closed, then we have, by Equalities (7) and (8),

$$\mathbb{A}_{\mathfrak{k}}^1 = \{\zeta_{s, \kappa} : s \in \mathfrak{k}^*; \kappa \in [0, 1)\} \cup \{\zeta_{0_{\mathfrak{k}}, \tau} : \tau \in \mathbb{R}_+\}. \quad (9)$$

Now, consider \mathfrak{k}_1 and \mathfrak{k}_2 to be two (not necessarily algebraically closed) fields equipped with the trivial norms. If the cardinalities of $\mathfrak{k}_1[\mathbf{t}]_{\text{irr}}$ and $\mathfrak{k}_2[\mathbf{t}]_{\text{irr}}$ are the same, then Proposition 2.1 implies that $\mathbb{A}_{\mathfrak{k}_1}^1$ is homeomorphic to $\mathbb{A}_{\mathfrak{k}_2}^1$. Conversely, suppose that we have a homeomorphism $\Psi : \mathbb{A}_{\mathfrak{k}_1}^1 \rightarrow \mathbb{A}_{\mathfrak{k}_2}^1$. For any $\mathbf{q}_1 \in \mathfrak{k}_1[\mathbf{t}]_{\text{irr}}$, it is easy to see that Ψ will send $\gamma_{\mathbf{q}_1, 0}$ to $\gamma_{\mathbf{q}_2, 0}$ for some $\mathbf{q}_2 \in \mathfrak{k}_2[\mathbf{t}]_{\text{irr}}$ (and vice-versa), because elements of the form $\gamma_{\mathbf{q}, 0}$ are all the free end points of maximal line-segments of $\mathbb{A}_{\mathfrak{k}_i}^1$ ($i = 1, 2$). Hence, we have a bijection between $\mathfrak{k}_1[\mathbf{t}]_{\text{irr}}$ and $\mathfrak{k}_2[\mathbf{t}]_{\text{irr}}$. These produce part (a) of the following corollary. The other parts of this corollary follow from part (a) and some well-known facts.

Corollary 2.2. *Suppose that \mathfrak{k}_1 and \mathfrak{k}_2 are two fields equipped with the trivial norm.*

(a) $\mathbb{A}_{\mathfrak{k}_1}^1$ and $\mathbb{A}_{\mathfrak{k}_2}^1$ are homeomorphic if and only if the cardinality of $\mathfrak{k}_1[\mathbf{t}]_{\text{irr}}$ equals that of $\mathfrak{k}_2[\mathbf{t}]_{\text{irr}}$.

(b) If both \mathfrak{k}_1 and \mathfrak{k}_2 are infinite (as sets), then $\mathbb{A}_{\mathfrak{k}_1}^1$ and $\mathbb{A}_{\mathfrak{k}_2}^1$ are homeomorphic if and only if \mathfrak{k}_1 and \mathfrak{k}_2 has the same cardinality.

(c) If \mathfrak{k}_1 is the algebraic closure of \mathfrak{k}_1 (again, endowed with the trivial norm), then $\mathbb{A}_{\mathfrak{k}_1}^1$ is homeomorphic to $\mathbb{A}_{\mathfrak{k}}^1$.

In the case when the complete valuation ring R is a field, then $|\cdot|$ is a trivial norm, and Proposition 2.1 gives the full description of \mathbb{A}_R^1 .

From now on, we consider the case when R is not a field; or equivalently, $|\cdot|$ is non-trivial.

Let us start with the following simple fact.

Lemma 2.3. (a) Suppose that S is an integral domain. If $P \subseteq S[\mathbf{t}]$ is a prime ideal with $I := P \cap S$ being non-zero, then one can find a prime ideal J (not necessarily non-zero) of $(S/I)[\mathbf{t}]$ such that $P = \Upsilon^{-1}(J)$, where $\Upsilon : S[\mathbf{t}] \rightarrow (S/I)[\mathbf{t}]$ is the quotient map.

(b) Suppose that $P \subseteq R[\mathbf{t}]$ is a non-zero prime ideal. Then $P \cap R \neq \{0_R\}$ if and only if $P = \tilde{Q}^{-1}(J)$ for a (not necessarily non-zero) prime ideal J of $F[\mathbf{t}]$.

In fact, as $\ker \Upsilon = I[\mathbf{t}] \subseteq P$, we know that $P = \Upsilon^{-1}(\Upsilon(P))$ and $\Upsilon(P)$ is a prime ideal of $(S/I)[\mathbf{t}]$. Furthermore, part (b) follows from part (a) and the fact that the only non-zero prime ideal of R is its maximal ideal (see e.g. Propositions 6 and 7 of [4, §VI.4.5]).

Proposition 2.4. Let R be a complete valuation ring, which is not a field. Under the notations as in the Introduction, one has

$$\mathbb{A}_R^1 = Q^\mathbb{A}(\mathbb{A}_F^1) \cup \bigcup_{\omega \in [1, \infty)} J_\omega^\mathbb{A}(\mathbb{A}_{K^\omega}^1).$$

Proof. We have already seen that $Q^\mathbb{A}(\mathbb{A}_F^1) \cup \bigcup_{\omega \in [1, \infty)} J_\omega^\mathbb{A}(\mathbb{A}_{K^\omega}^1) \subseteq \mathbb{A}_R^1$. For the other inclusion, let us pick an arbitrary element $\lambda \in \mathbb{A}_R^1$.

Suppose that $\ker |\cdot|_\lambda \cap R \neq \{0_R\}$. Then Lemma 2.3(b) gives a unique prime ideal $J \subseteq F[\mathbf{t}]$ with $\ker |\cdot|_\lambda = \tilde{Q}^{-1}(J)$. From this, one concludes that $\lambda = \mu \circ \tilde{Q}$ for an element $\mu \in \mathbb{A}_F^1$, i.e. $\lambda \in Q^\mathbb{A}(\mathbb{A}_F^1)$.

Suppose that $\ker |\cdot|_\lambda \cap R = \{0_R\}$. Let us extend $\lambda|_R$ to a function $|\cdot|_{\bar{\lambda}} : K \rightarrow \mathbb{R}_+$ by setting $|r|_{\bar{\lambda}} := |r^{-1}|_\lambda^{-1}$ whenever $r \in K \setminus R$. It is not hard to check that $|\cdot|_{\bar{\lambda}}$ is a multiplicative norm on K . For any $r \in K$, one has $|r| \leq 1$ if and only if $|r|_{\bar{\lambda}} \leq 1$. Thus, $|\cdot|_{\bar{\lambda}}$ is equivalent to $|\cdot|$, and one can find $\omega \in \mathbb{R}_+$ satisfying $|\cdot|_{\bar{\lambda}} = |\cdot|^\omega$ (see e.g. Proposition 3 of [4, §VI.3.2]). Moreover, as $|a|_{\bar{\lambda}} \leq |a|$ ($a \in R$), we know that $\omega \geq 1$. Finally, if we put

$$|a^{-1}\mathbf{p}|_{\bar{\lambda}} := |a|_{\bar{\lambda}}^{-1}|\mathbf{p}|_{\bar{\lambda}} \quad (a \in R^\times; \mathbf{p} \in R[\mathbf{t}]),$$

then $|\cdot|_{\bar{\lambda}}$ is a well-defined multiplicative seminorm on $K[\mathbf{t}]$ with $|r|_{\bar{\lambda}} = |r|_{\bar{\lambda}}$ ($r \in K$). Thus, $\bar{\lambda} \in \mathbb{A}_{K^\omega}^1$ and $\lambda = J_\omega^\mathbb{A}(\bar{\lambda})$. \square

It is clear that both $Q^\mathbb{A} : \mathbb{A}_F^1 \rightarrow \mathbb{A}_R^1$ and $J_\omega^\mathbb{A} : \mathbb{A}_{K^\omega}^1 \rightarrow \mathbb{A}_R^1$ ($\omega \in [1, \infty)$) are homeomorphisms onto their images (by considering the compact spaces $\mathcal{M}(F\{n^{-1}\mathbf{t}\})$ and $\mathcal{M}(K^\omega\{n^{-1}\mathbf{t}\})$ for all $n \in \mathbb{N}$). If $\lambda \in \mathbb{A}_K^1$ and $\omega \in [1, \infty)$, let us define $\lambda^\omega \in \mathbb{A}_{K^\omega}^1$ by

$$|\mathbf{p}|_{\lambda^\omega} := |\mathbf{p}|_\lambda^\omega \quad (\mathbf{p} \in K[\mathbf{t}]).$$

Obviously, $(\lambda, \omega) \mapsto J_\omega^\mathbb{A}(\lambda^\omega)$ induces a continuous bijection

$$\Lambda : \mathbb{A}_K^1 \times [1, \infty) \rightarrow \bigcup_{\omega \in [1, \infty)} J_\omega^\mathbb{A}(\mathbb{A}_{K^\omega}^1). \quad (10)$$

Later on, we will verify that Λ is actually a homeomorphism. We will also describe how the subspaces $Q^\mathbb{A}(\mathbb{A}_{K^\omega}^1)$ and $\bigcup_{\omega \in [1, \infty)} J_\omega^\mathbb{A}(\mathbb{A}_{K^\omega}^1)$ sit together in \mathbb{A}_R^1 .

From now on, we will identify \mathbb{A}_F^1 as subspaces of \mathbb{A}_R^1 , and may sometimes ignore the map $Q^{\mathbb{A}}$ if no confusion arises.

Lemma 2.5. *Suppose that F is algebraically closed. Consider a net $\{(\lambda_i, \omega_i)\}_{i \in \mathcal{I}}$ in $\mathbb{A}_K^1 \times [1, \infty)$.*

(a) *If $\{\lambda_i^{\omega_i}\}_{i \in \mathcal{I}}$ converges to an element in \mathbb{A}_F^1 , then $\lim_i \omega_i = \infty$ and $\limsup_{i \in \mathcal{I}} |\mathbf{t}|_{\lambda_i} \leq 1$.*

(b) *If $\lambda_i \in \mathbb{U}_1^K$ for all $i \in \mathcal{I}$ (see (2)), $\lim_i \omega_i = \infty$ and $\lim_i |\mathbf{t}|_{\lambda_i}^{\omega_i} = \tau \in [0, 1]$, then $\lambda_i^{\omega_i} \rightarrow \zeta_{0_F, \tau}$.*

Proof. (a) By (9) and the assumption, one can find $(x, \tau) \in F^* \times [0, 1) \cup \{0_F\} \times \mathbb{R}_+$ such that $\lambda_i^{\omega_i} \rightarrow \zeta_{x, \tau}$. Assume on the contrary that $\{\omega_i\}_{i \in \mathcal{I}}$ has a bounded subnet. Then there is a subnet $\{\omega_{i_j}\}_{j \in J}$ of $\{\omega_i\}_{i \in \mathcal{I}}$ converging to a number $\omega_0 \in [1, \infty)$. For any $a \in R$ with $|a| < 1$, one gets from

$$|a|^{\omega_{i_j}} = |a|_{\lambda_{i_j}^{\omega_{i_j}}} \rightarrow |Q(a)|_{\zeta_{x, \tau}} = 0,$$

that $|a|^{\omega_0} = 0$, and this contradicts the non-trivial assumption of $|\cdot|$. Hence, $\omega_i \rightarrow \infty$. On the other hand, since $\{|\mathbf{t}|_{\lambda_i}^{\omega_i}\}_{i \in \mathcal{I}}$ converges to $|\tilde{Q}(\mathbf{t})|_{\zeta_{x, \tau}}$, one concludes that $\limsup_{i \in \mathcal{I}} |\mathbf{t}|_{\lambda_i} \leq 1$, because otherwise, there exist $r > 1$ and a subnet $\{\lambda_{i_j}\}_{j \in \mathcal{J}}$ with $|\mathbf{t}|_{\lambda_{i_j}} \geq r$, which produces the contradiction that $|\mathbf{t}|_{\lambda_{i_j}}^{\omega_{i_j}} \geq r^{\omega_{i_j}} \rightarrow \infty$.

(b) If $b \in R$ with $|b| = 1$, then $|\mathbf{t} - b|_{\lambda_i}^{\omega_i} = 1$ (because $|\mathbf{t}|_{\lambda_i} < 1$) for all $i \in \mathcal{I}$. On the other hand, consider a polynomial $\mathbf{q} \in \ker \tilde{Q}$. There exist $a_0, \dots, a_n \in \ker Q$ with $\mathbf{q} = \sum_{k=0}^n a_k \mathbf{t}^k$. Hence,

$$|\mathbf{q}|_{\lambda_i} \leq \max_{k=0, \dots, n} |a_k| |\mathbf{t}|_{\lambda_i}^k \leq \max_{k=0, \dots, n} |a_k| < 1,$$

and one has $|\mathbf{q}|_{\lambda_i}^{\omega_i} \leq \max_{k=0, \dots, n} |a_k|^{\omega_i} \rightarrow 0$ (along i).

Now, let $\mathbf{p} \in R[\mathbf{t}]^*$ and $\{x_1, \dots, x_m\}$ be all the non-zero roots of $\tilde{Q}(\mathbf{p})$ in F (counting multiplicity). Pick $b_1, \dots, b_m \in R$ with $Q(b_l) = x_l$ ($l = 1, \dots, m$). Then $|b_l| = 1$ for all $l \in \{1, \dots, m\}$, and one can find $k \in \mathbb{Z}_+$ as well as $\mathbf{q}_0 \in \ker \tilde{Q}$ satisfying

$$\mathbf{p} = \mathbf{t}^k \cdot (\mathbf{t} - b_1) \cdots (\mathbf{t} - b_m) + \mathbf{q}_0.$$

The above and the hypothesis will then tell us that $|\mathbf{t}^k \cdot (\mathbf{t} - b_1) \cdots (\mathbf{t} - b_m)|_{\lambda_i}^{\omega_i} \rightarrow \tau^k$ and $|\mathbf{q}_0|_{\lambda_i}^{\omega_i} \rightarrow 0$. Consequently, $|\mathbf{p}|_{\lambda_i}^{\omega_i} \rightarrow \tau^k = |\tilde{Q}(\mathbf{p})|_{\zeta_{0_F, \tau}}$ as is required (observe that $0 \leq \tau \leq 1$). \square

The following is our first main theorem that gives a full description of the topological space \mathbb{A}_R^1 .

Theorem 2.6. *Let R be a complete valuation ring which is not a field, F be the residue field of R , and K be the field of fractions of R equipped with the induced absolute value $|\cdot|$. Suppose that K is algebraically closed. We fix a cross section $\tilde{F} \subseteq R$ of $Q : R \rightarrow F$ that contains 0_R .*

(a) \mathbb{A}_F^1 is closed in \mathbb{A}_R^1 .

(b) The map $\Lambda : \mathbb{A}_K^1 \times [1, \infty) \rightarrow \bigcup_{\omega \in [1, \infty)} J_{\omega}^{\mathbb{A}}(\mathbb{A}_{K^{\omega}}^1)$ in (10) is a homeomorphism.

(c) Suppose that $\{(\lambda_i, \omega_i)\}_{i \in \mathcal{I}}$ is a net in $\mathbb{A}_K^1 \times [1, \infty)$. Then $\{\lambda_i^{\omega_i}\}_{i \in \mathcal{I}}$ converges to an element $\lambda_0 \in \mathbb{A}_F^1$ if and only if $\omega_i \rightarrow \infty$ and either one of the following holds:

C1). there exist $\tau_1 \in [0, 1)$ and $b \in R$ such that $|\mathbf{t} - b|_{\lambda_i}^{\omega_i} \rightarrow \tau_1$ (in this case, $\lambda_0 = \zeta_{Q(b), \tau_1}$);

C2). one can find $\tau_2 \in (1, \infty)$ such that $|\mathbf{t}|_{\lambda_i}^{\omega_i} \rightarrow \tau_2$ (in this case, $\lambda_0 = \zeta_{0_F, \tau_2}$);

C3). $|\mathbf{t} - c|_{\lambda_i}^{\omega_i} \rightarrow 1$ for any $c \in \tilde{F}$ (in this case, $\lambda_0 = \zeta_{0_F, 1}$).

(d) Under the homeomorphism in part (b), one may regard $\mathbb{A}_K^1 \times [1, \infty)$ as an open dense subset of \mathbb{A}_R^1 . Consequently, the image of $K \times (1, \infty)$ is dense in \mathbb{A}_R^1 .

Proof. The non-triviality of $|\cdot|$ produces $a_0 \in R$ with $|a_0| \in (0, 1)$. Moreover, since K is algebraically closed, so is F . As in (9), one has

$$\mathbb{A}_F^1 = \{\zeta_{x, \tau} : (x, \tau) \in F \times [0, 1) \cup \{0_F\} \times [1, \infty)\}.$$

(a) Suppose on the contrary that there is a net $\{(x_i, \tau_i)\}_{i \in \mathcal{I}}$ in $F \times [0, 1) \cup \{0_F\} \times [1, \infty)$ with $\zeta_{x_i, \tau_i} \rightarrow \lambda^\omega$ for some $(\lambda, \omega) \in \mathbb{A}_K^1 \times [1, \infty)$. Then, $0 = |Q(a_0)|_{\zeta_{x_i, \tau_i}} \rightarrow |a_0|^\omega$, which is absurd.

(b) As said in the paragraph preceding Lemma 2.5, the map Λ is a continuous bijection. If $\{(\lambda_i, \omega_i)\}_{i \in \mathcal{I}}$ is a net in $\mathbb{A}_K^1 \times [1, \infty)$ satisfying $\lambda_i^{\omega_i} \rightarrow \lambda_0^{\omega_0}$ for some $(\lambda_0, \omega_0) \in \mathbb{A}_K^1 \times [1, \infty)$, then

$$|a_0|^{\omega_i/\omega_0} = |a_0|_{\lambda_i^{\omega_i}}^{1/\omega_0} \rightarrow |a_0|$$

and so $\omega_i \rightarrow \omega_0$. From this, one can also deduce that $|\mathbf{p}|_{\lambda_i} \rightarrow |\mathbf{p}|_{\lambda_0}$ for every $\mathbf{p} \in R[\mathbf{t}]$. Consequently, the inverse of Λ is continuous.

(c) \Rightarrow . Suppose that $\{\lambda_i^{\omega_i}\}_{i \in \mathcal{I}}$ converges to $\zeta_{Q(b), \tau_1}$ for some $b \in R$ and $\tau_1 \in [0, 1)$. Then we have $\omega_i \rightarrow \infty$ (by Lemma 2.5(a)) and $|\mathbf{t} - b|_{\lambda_i}^{\omega_i} \rightarrow |\mathbf{t} - Q(b)|_{\zeta_{Q(b), \tau_1}} = \tau_1$. This verifies Condition (C1).

On the other hand, suppose that $\{\lambda_i^{\omega_i}\}_{i \in \mathcal{I}}$ converges to $\zeta_{0_F, \kappa}$ for some $\kappa \in [1, \infty)$. Again, one has $\omega_i \rightarrow \infty$ because of Lemma 2.5(a). Moreover, as $\kappa \geq 1$, one knows that

$$|\mathbf{t} - c|_{\lambda_i}^{\omega_i} \rightarrow |\mathbf{t} - Q(c)|_{\zeta_{0_F, \kappa}} = \kappa \quad (c \in R).$$

Hence, either Conditions (C2) or (C3) holds (depending on whether $\kappa = 1$).

\Leftarrow . Suppose that $\omega_i \rightarrow \infty$ and Condition (C1) holds. Then $|\mathbf{t}|_{\lambda_i - b}^{\omega_i} = |\mathbf{t} - b|_{\lambda_i}^{\omega_i} \rightarrow \tau_1 < 1$ (see (4)), which implies that $|\mathbf{t}|_{\lambda_i - b} < 1$ eventually. By Lemma 2.5(b), we know that $(\lambda_i - b)^{\omega_i} \rightarrow \zeta_{0_F, \tau_1}$ and hence $\lambda_i^{\omega_i} \rightarrow \zeta_{Q(b), \tau_1}$.

Secondly, suppose that $\omega_i \rightarrow \infty$ and Condition (C2) holds. We first note that

$$\limsup_{i \in \mathcal{I}} |\mathbf{t}|_{\lambda_i} \leq 1, \quad (11)$$

because otherwise, one can find a subnet such that $|\mathbf{t}|_{\lambda_{i_j}}^{\omega_{i_j}} \rightarrow \infty$. Let us also show that

$$|\mathbf{t} - c'|_{\lambda_i}^{\omega_i} \rightarrow \tau_2 \quad (c' \in R). \quad (12)$$

In fact, as $\tau_2 > 1$, when i is large, one has $|\mathbf{t}|_{\lambda_i} > 1$, and hence $|\mathbf{t} - c'|_{\lambda_i} = |\mathbf{t}|_{\lambda_i}$ ($c' \in R$) and Relation (12) follows.

Now, for any $\mathbf{q} \in \ker \tilde{Q}$, we let $a_0, \dots, a_n \in \ker Q$ be the elements with $\mathbf{q} = \sum_{k=0}^n a_k \mathbf{t}^k$. As $\kappa := \max\{|a_0|, \dots, |a_n|\} < 1$, we obtain from (11) an element $i_0 \in \mathcal{I}$ satisfying $\sup_{i \geq i_0} |\mathbf{t}|_{\lambda_i} < 1/\kappa^{\frac{1}{n+1}}$ and hence for $i \geq i_0$,

$$|\mathbf{q}|_{\lambda_i} \leq \max_{k=0, \dots, n} |a_k| |\mathbf{t}|_{\lambda_i}^k < \kappa^{\frac{1}{n+1}}.$$

This means that $|\mathbf{q}|_{\lambda_i}^{\omega_i} \rightarrow 0$. Consider now a polynomial $\mathbf{p} \in R[\mathbf{t}]$ of degree m . Let $\{y_1, \dots, y_m\}$ be all the roots of $\tilde{Q}(\mathbf{p})$ in F (counting multiplicity) and let c_1, \dots, c_m be elements in R satisfying $Q(c_l) = y_l$ ($l = 1, \dots, m$). Then one can find $\mathbf{q}_0 \in \ker \tilde{Q}$ with

$$\mathbf{p} = (\mathbf{t} - c_1) \cdots (\mathbf{t} - c_m) + \mathbf{q}_0.$$

Thus, (12) and the above imply that $|\mathbf{p}|_{\lambda_i}^{\omega_i} \rightarrow \tau_2^m = \zeta_{0_F, \tau_2}(\tilde{Q}(\mathbf{p}))$ (notice that $\tau_2 \geq 1$).

Finally, we consider the case when $\omega_i \rightarrow \infty$ and Condition (C3) holds. As $0_R \in \tilde{F}$, we know that $|\mathbf{t}|_{\lambda_i}^{\omega_i} \rightarrow 1$ and Relation (11) is satisfied. Moreover, we have $|\mathbf{t} - c'|_{\lambda_i}^{\omega_i} \rightarrow 1$ ($c' \in R$). Indeed, if $c' \in R$, one can find $c \in \tilde{F}$ satisfying $|c' - c| < 1$. As $\omega_i \rightarrow \infty$, there exists $i_0 \in \mathcal{I}$ such that for any $i \geq i_0$, one has both $|c' - c|^{\omega_i} < 1/2$ as well as $|\mathbf{t} - c|_{\lambda_i}^{\omega_i} > 1/2$, and hence

$$|\mathbf{t} - c'|_{\lambda_i} = |\mathbf{t} - c|_{\lambda_i} \quad (i \geq i_0).$$

This implies that $|\mathbf{t} - c'|_{\lambda_i}^{\omega_i} \rightarrow 1$ as required. Now, it follows from the same argument as that for Condition (C2) that $|\mathbf{p}|_{\lambda_i}^{\omega_i} \rightarrow 1 = \zeta_{0_F, 1}(\tilde{Q}(\mathbf{p}))$, for any $\mathbf{p} \in R[\mathbf{t}]$.

(d) If $\tau_1 < 1$, then it follows easily from part (c) that $\zeta_{b, \tau_1}^n \rightarrow \zeta_{Q(b), \tau_1}$ for every $b \in R$. If $\tau_2 > 1$, it is not hard to see from part (c) that $\zeta_{0_K, \tau_2}^n \rightarrow \zeta_{0_F, \tau_2}$. Furthermore, since $\zeta_{0_K, 1}(\mathbf{t} - c) = 1$ for all

$c \in R$, we know from part (c) that $\zeta_{0_K,1}^n \rightarrow \zeta_{0_F,1}$. These establish the first statement. The second statement follows from the first one, part (b) as well as the fact that K is dense in \mathbb{A}_K^1 . \square

From now on, we will also identify $\mathbb{A}_K^1 \times [1, \infty)$ with $\bigcup_{\omega \in [1, \infty)} J_\omega^\mathbb{A}(\mathbb{A}_{K^\omega}^1)$ (i.e. (λ, ω) is identified with λ^ω) as well as $K \times [1, \infty)$ with its image in $\bigcup_{\omega \in [1, \infty)} J_\omega^\mathbb{A}(\mathbb{A}_{K^\omega}^1)$ (i.e. (r, ω) is identified with $\zeta_{r,0}^\omega$).

Remark 2.7. (a) In the proof of Theorem 2.6(c), we see that $\lambda_i^{\omega_i} \rightarrow \zeta_{0_F, \kappa}$ for some $\kappa \in [1, \infty)$ if and only if $\omega_i \rightarrow \infty$ and $|\mathbf{t} - c'|_{\lambda_i}^{\omega_i} \rightarrow \kappa$, for all $c' \in R$.

(b) Unlike the case of $\tau_2 \in (1, \infty)$, the requirement $|\mathbf{t}|_{\lambda_i}^{\omega_i} \rightarrow 1$ does not imply $\lambda_i^{\omega_i} \rightarrow \zeta_{0_F, 1}$. For example, if we set $\lambda_i := \zeta_{1_K, 0}$ ($i \in \mathfrak{I}$), then $|\mathbf{t}|_{\lambda_i}^{\omega_i} = 1$ but $|\mathbf{t} - 1|_{\lambda_i}^{\omega_i} = 0$ for all $i \in \mathfrak{I}$.

(c) Consider the uniform structure on $K \times (1, \infty)$ that is induced by the fundamental system of entourages of the form:

$$\{((s_1, \omega_1), (s_2, \omega_2)) : ||\mathbf{p}(s_1)|^{\omega_1} - |\mathbf{p}(s_2)|^{\omega_2}|| < \epsilon, \text{ for each } \mathbf{p} \in X\},$$

where ϵ runs through all positive numbers and X runs through all non-empty finite subsets of $R[\mathbf{t}]$. As a uniform space, \mathbb{A}_R^1 is the completion of $K \times (1, \infty)$ under this structure (because of Theorem 2.6(d)).

Before continuing our discussion, let us set some more notations. For any $\mathbf{p} \in R[\mathbf{t}]$ as well as $\tau, \epsilon \in \mathbb{R}_+$, we denote

$$U_{\tau, \epsilon}^{\mathbf{p}} := \{\mu \in \mathbb{A}_R^1 : \tau - \epsilon < \mu(\mathbf{p}) < \tau + \epsilon\}.$$

It follows from the definition that $U_{\tau, \epsilon}^{\mathbf{p}}$ an open subset of \mathbb{A}_R^1 .

Proposition 2.4 as well as parts (a) and (b) of Theorem 2.6 tell us that \mathbb{A}_R^1 contains the product space $\mathbb{A}_K^1 \times [1, \infty)$ as an open subset with its complement being \mathbb{A}_F^1 . Moreover, by Lemma 2.5(a), we know that for every $\kappa \geq 1$, the subset $\mathbb{A}_K^1 \times [1, \kappa]$ is closed in \mathbb{A}_R^1 .

Since the topology on the product space $\mathbb{A}_K^1 \times [1, \infty)$ is well-known and Proposition 2.1 describes the topological space \mathbb{A}_F^1 , the topology of \mathbb{A}_R^1 can be determined if one knows the description of neighborhood bases over elements in \mathbb{A}_F^1 . Through Theorem 2.6, these bases will be described as follows:

N1) Consider $\tau \in [0, 1)$ and $b \in \tilde{F}$ (see Theorem 2.6). If $\kappa \geq 1$ and $0 < \epsilon < 1 - \tau$, one can check easily, using Relation (9), that

$$U_{\tau, \epsilon}^{\mathbf{t}-b} \setminus \mathbb{A}_K^1 \times [1, \kappa] = \{(\lambda, \omega) \in \mathbb{A}_K^1 \times [1, \infty) : \omega > \kappa; ||\mathbf{t} - b|_\lambda^\omega - \tau| < \epsilon\} \cup \{\zeta_{Q(b), v} : v \in [0, 1); |v - \tau| < \epsilon\}.$$

Thus, by Theorem 2.6(c) and Proposition 2.1, the countable collection:

$$\{U_{\tau, 1/n}^{\mathbf{t}-b} \setminus \mathbb{A}_K^1 \times [1, m] : m, n \in \mathbb{N} \text{ such that } 1/n \in (0, 1 - \tau)\}$$

constitutes an open neighborhood basis of $\zeta_{Q(b), \tau}$.

N2) Consider $\tau > 1$. If $\kappa \geq 1$ and $0 < \epsilon < \tau - 1$, one may verify that

$$U_{\tau, \epsilon}^{\mathbf{t}} \setminus \mathbb{A}_K^1 \times [1, \kappa] = \{(\lambda, \omega) \in \mathbb{A}_K^1 \times [1, \infty) : \omega > \kappa; ||\mathbf{t}|_\lambda^\omega - \tau| < \epsilon\} \cup \{\zeta_{0_F, v} : \tau - \epsilon < v < \tau + \epsilon\}.$$

Again, Theorem 2.6(c) and Proposition 2.1 imply that the countable collection:

$$\{U_{\tau, 1/n}^{\mathbf{t}} \setminus \mathbb{A}_K^1 \times [1, m] : m, n \in \mathbb{N} \text{ such that } 1/n < \tau - 1\}$$

constitutes an open neighborhood basis of $\zeta_{0_F, \tau}$.

N3) Let \mathcal{F} be the collection of all finite subsets of \tilde{F} such that each of them contains 0_R . If $X \in \mathcal{F}$, $\kappa \geq 1$ and $\epsilon \in (0, 1)$, it is not hard to see that

$$\begin{aligned} \bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c} \setminus \mathbb{A}_K^1 \times [1, \kappa] = \\ \{(\lambda, \omega) \in \mathbb{A}_K^1 \times [1, \infty) : \omega > \kappa; |\mathbf{t} - c|_\lambda^\omega - 1| < \epsilon, \text{ for any } c \in X\} \cup \\ \{\zeta_{Q(c),v} : c \in X; v \in (1 - \epsilon, 1)\} \cup \{\zeta_{0_F,v} : v \in [1, 1 + \epsilon)\} \cup \{\zeta_{Q(b),v} : b \in \tilde{F} \setminus X; v \in [0, 1)\}. \end{aligned}$$

It now follows from Theorem 2.6(c) and Proposition 2.1 that the collection:

$$\left\{ \bigcap_{c \in X} U_{1,1/n}^{\mathbf{t}-c} \setminus \mathbb{A}_K^1 \times [1, m] : m, n \in \mathbb{N}; X \in \mathcal{F} \right\}$$

constitutes an open neighborhood basis of $\zeta_{0_F,1}$.

One may use the above information to obtain certain topological properties of \mathbb{A}_R^1 . The following is an illustration.

Theorem 2.8. *Let R be a complete valuation ring which is not a field, with its field of fractions being algebraically closed. The following properties hold.*

- (a) \mathbb{A}_R^1 is path connected.
- (b) \mathbb{A}_R^1 is locally path connected.
- (c) \mathbb{A}_R^1 is first countable if and only if F is countable and \mathbb{A}_K^1 is first countable.

Proof. (a) By Proposition 2.1, \mathbb{A}_F^1 is path connected. Moreover, as said in the Introduction, \mathbb{A}_K^1 is path connected, and this gives the path connectedness of $\mathbb{A}_K^1 \times [1, \infty)$. Consider $(x_0, \tau_0) \in F \times [0, 1) \cup \{0_F\} \times [1, \infty)$ and $(\lambda_0, \omega_0) \in \mathbb{A}_K^1 \times [1, \infty)$. As in the proof of Theorem 2.6(d), one has $\zeta_{0_K,0}^\omega \rightarrow \zeta_{0_F,0}$ (when $\omega \rightarrow \infty$), and this produces a path joining $\zeta_{0_K,0}$ to $\zeta_{0_F,0}$. By considering a path in \mathbb{A}_F^1 (respectively, $\mathbb{A}_K^1 \times [1, \infty)$) joining ζ_{x_0,τ_0} to $\zeta_{0_F,0}$ (respectively, joining $\lambda_0^{\omega_0}$ to $\zeta_{0_K,0}$), one obtains a path that joins ζ_{x_0,τ_0} to $\lambda_0^{\omega_0}$.

(b) As recalled in the Introduction, \mathbb{A}_K^1 is locally path connected, and hence so is the open subset $\mathbb{A}_K^1 \times [1, \infty)$ of \mathbb{A}_R^1 .

Let us now verify the path connectedness of open sets of the form as in (N1). Let $\tau \in [0, 1)$ and $b \in \tilde{F}$. Fix $\kappa \geq 1$ as well as $\epsilon \in (0, 1 - \tau)$. For simplicity, we denote

$$V^\kappa := \mathbb{A}_K^1 \times (\kappa, \infty).$$

It is easy to see that $\{\zeta_{Q(b),v} : v \in [0, 1); |v - \tau| < \epsilon\}$ is path connected (see Proposition 2.1). Thus, in order to verify the path connectedness of $U_{\tau,\epsilon}^{\mathbf{t}-b} \setminus \mathbb{A}_K^1 \times [1, \kappa]$, it suffices to show that an arbitrary fixed element $\lambda_0^{\omega_0} \in U_{\tau,\epsilon}^{\mathbf{t}-b} \cap V^\kappa$ can be joined to $\zeta_{Q(b),\tau}$ through a path inside $U_{\tau,\epsilon}^{\mathbf{t}-b} \setminus \mathbb{A}_K^1 \times [1, \kappa]$.

Notice that the subset

$$\{\lambda \in \mathbb{A}_K^1 : |\mathbf{t} - b|_\lambda^{\omega_0} - \tau| < \epsilon\}$$

is open in \mathbb{A}_K^1 and contains $\lambda_0^{\omega_0}$. Hence, by the locally path connectedness of \mathbb{A}_K^1 and the density of the image of K in \mathbb{A}_K^1 , one may assume that $\lambda_0 = \zeta_{s_0,0}$ for an element $s_0 \in K$. The requirement of $\zeta_{s_0,0}^{\omega_0} \in U_{\tau,\epsilon}^{\mathbf{t}-b}$ implies that

$$\tau - \epsilon < |s_0 - b|^{\omega_0} < \tau + \epsilon.$$

For every $v \in [0, |s_0 - b|]$, the relation

$$|\mathbf{t} - b|_{\zeta_{s_0,v}^{\omega_0}} = \max\{v, |s_0 - b|\}^{\omega_0} = |s_0 - b|^{\omega_0} \in (\tau - \epsilon, \tau + \epsilon)$$

tells us that $\zeta_{s_0,v}^{\omega_0} \in U_{\tau,\epsilon}^{\mathbf{t}-b}$. Thus, $\{\zeta_{s_0,v}^{\omega_0} : v \in [0, |s_0 - b|]\}$ is a path in $U_{\tau,\epsilon}^{\mathbf{t}-b}$ joining $\zeta_{s_0,0}^{\omega_0}$ to $\zeta_{s_0,|s_0-b|}^{\omega_0} = \zeta_{b,|s_0-b|}$.

Similarly, the path $\{\zeta_{b,v}^{\omega_0} : v \text{ is in between } |s_0 - b| \text{ and } \tau^{1/\omega_0}\}$ that joins $\zeta_{b,|s_0-b|}^{\omega_0}$ to $\zeta_{b,\tau^{1/\omega_0}}^{\omega_0}$ also lies inside $U_{\tau,\epsilon}^{\mathbf{t}-b}$. Moreover, it follows from

$$|\mathbf{t} - b|_{\zeta_{b,\tau^{1/\omega_0}}^{\omega_0}} = \tau \quad (13)$$

and Theorem 2.6(c) that the net $\{\zeta_{b,\tau^{1/\omega_0}}^{\omega_0} : \omega \geq \omega_0\}$ converges to $\zeta_{Q(b),\tau}$ when $\omega \rightarrow \infty$. Therefore,

$$\{\zeta_{b,\tau^{1/\omega_0}}^{\omega_0} : \omega \geq \omega_0\} \cup \{\zeta_{Q(b),\tau}\}$$

is a path in \mathbb{A}_K^1 joining $\zeta_{b,\tau^{1/\omega_0}}^{\omega_0}$ to $\zeta_{Q(b),\tau}$. Furthermore, (13) also ensures that this path lies inside $U_{\tau,\epsilon}^{\mathbf{t}-b}$. Consequently, $U_{\tau,\epsilon}^{\mathbf{t}-b} \setminus \mathbb{A}_K^1 \times [1, \kappa]$ is path connected (note that ω_0 as well as all the ω in the above are strictly bigger than κ).

In the same way, for any fixed $\tau \in (1, \infty)$, one can establish the path connectedness of open sets as in (N2), i.e., $U_{\tau,\epsilon}^{\mathbf{t}} \setminus \mathbb{A}_K^1 \times [1, \kappa]$ for $\kappa \geq 1$ and $\epsilon \in (0, \tau - 1)$.

It remains to show that open sets of the form as in (N3), namely, the set $\bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c} \setminus \mathbb{A}_K^1 \times [1, \kappa]$ (for $X \in \mathcal{F}$, $\kappa \geq 1$ and $\epsilon \in (0, 1)$) is path connected. As in the above, we need to find a path in $\bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c} \setminus \mathbb{A}_K^1 \times [1, \kappa]$ that joins an arbitrary chosen element $\lambda_0^{\omega_0} \in \bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c} \cap V^\kappa$ to $\zeta_{0_F,1}$. Again, we may assume that $\lambda_0 = \zeta_{s_0,0}$ for an element $s_0 \in K$. The condition $\zeta_{s_0,0}^{\omega_0} \in \bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c}$ implies that

$$(1 - \epsilon)^{1/\omega_0} < |s_0 - c| < (1 + \epsilon)^{1/\omega_0} \quad (c \in X),$$

and, in particular, $|s_0|^{\omega_0} \in (1 - \epsilon, 1 + \epsilon)$ (because $0_R \in X$ by the assumption of \mathcal{F}). For every $v \in [0, |s_0|]$, the equality

$$|\mathbf{t} - c|_{\zeta_{s_0,v}^{\omega_0}} = \max\{v, |s_0 - c|\}^{\omega_0} \quad (c \in X) \quad (14)$$

will then ensure that $\zeta_{s_0,v}^{\omega_0} \in \bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c}$. In addition, a similar relation as (14) tells us that the path $\{\zeta_{0_K,v}^{\omega_0} : v \text{ is in between } |s_0| \text{ and } 1\}$ lies inside $\bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c}$. On the other hand, using Theorem 2.6(c), we see that $\{\zeta_{0_K,1}^{\omega_0} : \omega \geq \omega_0\}$ converges to $\zeta_{0_F,1}$ when $\omega \rightarrow \infty$, and this produces a path joining $\zeta_{0_K,1}^{\omega_0}$ to $\zeta_{0_F,1}$. Moreover, the equalities $\zeta_{0_K,1}(\mathbf{t} - c) = 1$ ($c \in X$) gives $\zeta_{0_K,1}^{\omega_0} \in \bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c}$ ($\omega \geq \omega_0$). Consequently, we obtain a path in $\bigcap_{c \in X} U_{1,\epsilon}^{\mathbf{t}-c} \setminus \mathbb{A}_K^1 \times [1, \kappa]$ joining $\lambda_0^{\omega_0}$ to $\zeta_{0_F,1}$.

(c) Clearly, the countability of F and the first countability of \mathbb{A}_K^1 will imply the first countability of \mathbb{A}_R^1 . Conversely, suppose that \mathbb{A}_R^1 is first countable. Then $\mathbb{A}_K^1 \times [1, \infty)$ is first countable and so is \mathbb{A}_K^1 . Moreover, there is a countable subcollection

$$\left\{ \bigcap_{c \in X_k} U_{1, \frac{1}{n_k}}^{\mathbf{t}-c} \setminus \mathbb{A}_K^1 \times [1, m_k] : k \in \mathbb{N} \right\}$$

that form a neighborhood basis for $\zeta_{0_F,1}$ in \mathbb{A}_R^1 . Let $C := \bigcup_{k \in \mathbb{N}} X_k$. We know that whenever a net $\{(\lambda_i, \omega_i)\}_{i \in \mathbb{I}}$ in $\mathbb{A}_K^1 \times [1, \infty)$ satisfying $\omega_i \rightarrow \infty$ as well as $|\mathbf{t} - b|_{\lambda_i}^{\omega_i} \rightarrow 1$ ($b \in C$), we have $\lambda_i^{\omega_i} \rightarrow \zeta_{0_F,1}$. Assume on the contrary that \tilde{F} is uncountable, then there exists $c_0 \in \tilde{F} \setminus C$ (which implies $|c_0 - b| = 1$ for all $b \in C$). If we set $\omega_n := n$ and $\lambda_n := \zeta_{c_0,0}$ ($n \in \mathbb{N}$), then $|\mathbf{t} - b|_{\lambda_n}^{\omega_n} = 1$ ($n \in \mathbb{N}$), for each $b \in C$, but $|\mathbf{t} - c_0|_{\lambda_n}^{\omega_n} = 0$ ($n \in \mathbb{N}$), which contradicts $\lambda_n^{\omega_n} \rightarrow \zeta_{0_F,1}$. \square

Remark 2.9. Unlike \mathbb{A}_K^1 , the topological space \mathbb{A}_R^1 is far from having a \mathbb{R} -tree structure. Actually, for any connected open subset $V \subseteq \mathbb{A}_R^1$ and any $\lambda_1, \lambda_2 \in V$, there exist infinitely many paths in V joining λ_1 and λ_2 .

In fact, consider $i \in \{1, 2\}$. Let $U_i \subseteq V$ be a path connected open neighborhood of λ_i (see Theorem 2.8(b)) and fix any $(\mu_i, \kappa_i) \in \mathbb{A}_K^1 \times (1, \infty) \cap U_i$ (see Theorem 2.6). Thus, λ_i is joined to (μ_i, κ_i) through a path inside U_i . Choose a connected open neighborhood W_i of μ_i in \mathbb{A}_K^1 as well as a number $\epsilon \in (0, 1 - \kappa_i)$ such that $W_i \times (\kappa_i - \epsilon, \kappa_i + \epsilon) \subseteq U_i$. Pick any $\nu_i \in W_i$. There exist infinitely many

paths in $W_i \times (\kappa_i - \epsilon, \kappa_i + \epsilon)$ joining (μ_i, κ_i) to (ν_i, κ_i) . As V is path connected, there exists a path in V joining (ν_1, κ_1) to (ν_2, κ_2) . In this way, we obtain infinitely many paths joining λ_1 and λ_2 .

Similar to Theorem 2.8(c), there is also a description of the second countability of \mathbb{A}_R^1 . In fact, one has the following more general result. This result could be a known fact, but since we do not find an explicit reference for it, we present its simple argument here.

Proposition 2.10. *If $(S, \|\cdot\|)$ is a commutative unital Banach ring with multiplicative norm, the following statements are equivalent.*

- S1). \mathbb{A}_S^1 is second countable.
- S2). S is separable as a metric space.
- S3). $\mathcal{M}(S\{n^{-1}\mathbf{t}\})$ is metrizable for every $n \in \mathbb{N}$.

Proof. (S1) \Rightarrow (S2). For any $a \in S$, we defined $\zeta_a \in \mathbb{A}_S^1$ by $\zeta_a(\mathbf{p}) := \|\mathbf{p}(a)\|$ ($\mathbf{p} \in S[\mathbf{t}]$). It is easy to see that $a \mapsto \zeta_a$ is a homeomorphism from S onto its image in \mathbb{A}_S^1 . Thus, S is also second countable and hence is separable.

(S2) \Rightarrow (S3). Suppose that S contains a countable dense subset S_0 . We denote by \mathcal{F}_0 the collection of non-empty finite subsets of $S[\mathbf{t}]$ consisting of polynomials with coefficients in S_0 . For a fixed $n \in \mathbb{N}$, since $|\mathbf{t}|_\mu \leq n$ for any $\mu \in \mathcal{M}(S\{n^{-1}\mathbf{t}\})$, it is not hard to see that the countable family

$$\{E_{1/k}^X \cap \mathcal{M}(S\{n^{-1}\mathbf{t}\}) \times \mathcal{M}(S\{n^{-1}\mathbf{t}\}) : k \in \mathbb{N}; X \in \mathcal{F}_0\}$$

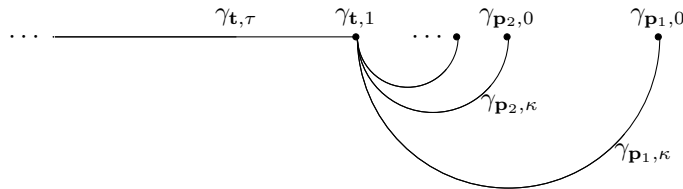
(see Relation (1) for the meaning of $E_{1/k}^X$) forms a fundamental system of entourages for the Berkovich uniform structure on $\mathcal{M}(S\{n^{-1}\mathbf{t}\})$. Consequently, the Berkovich uniform structure (and hence the topology defined by it) is pseudo-metrizable (see, e.g., Theorem 13 in chapter 6 of [6]). However, since the topology on $\mathcal{M}(S\{n^{-1}\mathbf{t}\})$ is Hausdorff, we conclude that this topology is indeed metrizable.

(S3) \Rightarrow (S1). Since $\mathcal{M}(S\{n^{-1}\mathbf{t}\})$ is a compact metric space, it is second countable, and so, the open subset $\mathbb{U}_n^S \subseteq \mathcal{M}(S\{n^{-1}\mathbf{t}\})$ (see (2)) is also second countable. Consequently, \mathbb{A}_S^1 is second countable. \square

Clearly, the Banach ring R is separable (equivalently, second countable) if and only if K is separable as a metric space. Note also that if a complete valued field is algebraically closed and spherically complete, then it is not separable (see e.g. [5, Remark 1.4]). However, the completion of the algebraic closure of a separable complete valued field is again separable (see e.g. [5, Remark 1.3]).

Example 2.11. Let \mathfrak{k} be a field equipped with the trivial norm. By Proposition 2.1, one can define a metric on $\mathbb{A}_{\mathfrak{k}}^1$ through the “geodesic distance” d_G of two given points. If \mathfrak{k} is uncountable, then Proposition 2.10 tells us that $\mathbb{A}_{\mathfrak{k}}^1$ is not metrizable, and hence the topology induced by d_G is different from the pointwise convergence topology on $\mathbb{A}_{\mathfrak{k}}^1$.

On the other hand, suppose that \mathfrak{k} is at most countable and $\mathbf{p}_1, \mathbf{p}_2, \dots$ are all the elements in $\mathfrak{k}[\mathbf{t}]_{\text{irr}}$. If we rescale the “geodesic distance” so that $d_G(\gamma_{\mathbf{p}_k, 0}, \gamma_{\mathbf{t}, 1}) \rightarrow 0$ when $k \rightarrow \infty$, then it is not hard to see that this metric defines the topology of pointwise convergence on $\mathbb{A}_{\mathfrak{k}}^1$. In this case, $\mathbb{A}_{\mathfrak{k}}^1$ is the following closed subspace of \mathbb{R}^2 :



In the following, we will have a look at the set of “type I points” of \mathbb{A}_R^1 , i.e. those multiplicative semi-norms $\lambda \in \mathbb{A}_R^1$ with $\ker |\cdot|_\lambda \neq \{0_R\}$.

Corollary 2.12. *Let R , $(K, |\cdot|)$, F and \tilde{F} be as in Theorem 2.6. Denote*

$$\mathbb{A}_{R,Z}^1 := \{\lambda \in \mathbb{A}_R^1 : \ker |\cdot|_\lambda \neq \{0_R\}\}.$$

(a) $\mathbb{A}_{R,Z}^1 = \mathbb{A}_F^1 \cup K \times [1, \infty)$ (and hence, $\mathbb{A}_{R,Z}^1$ is dense in \mathbb{A}_R^1).

(b) Suppose that $\{(s_i, \omega_i)\}_{i \in \mathcal{I}}$ is a net in $K \times [1, \infty)$.

- $\zeta_{s_i,0}^{\omega_i} \rightarrow \zeta_{Q(b),\tau_1}$ for some $b \in R$ and $\tau_1 \in [0, 1)$ if and only if $\omega_i \rightarrow \infty$ and $|s_i - b|^{\omega_i} \rightarrow \tau_1$;
- $\zeta_{s_i,0}^{\omega_i} \rightarrow \zeta_{0_F,\tau_2}$ for a number $\tau_2 \in (1, \infty)$ if and only if $\omega_i \rightarrow \infty$ and $|s_i|^{\omega_i} \rightarrow \tau_2$;
- $\zeta_{s_i,0}^{\omega_i} \rightarrow \zeta_{0_F,1}$ if and only if $\omega_i \rightarrow \infty$ and $|s_i - c|^{\omega_i} \rightarrow 1$, for any $c \in \tilde{F}$.

(c) If $\{(s_i, \omega_i)\}_{i \in \mathcal{I}}$ is a net in $K \times [1, \infty)$ such that $\zeta_{s_i,0}^{\omega_i} \rightarrow \zeta_{Q(b),\tau_1}$ for some $b \in R$ and $\tau_1 \in [0, 1)$, then s_i eventually belongs to the “open ball” of K of radius 1 and center b .

Proof. Since part (b) follows from Theorem 2.6 and part (c) follows directly from part (b), we will only establish part (a). In fact, it is clear that $\mathbb{A}_F^1 \cup K \times [1, \infty) \subseteq \mathbb{A}_{R,Z}^1$. Consider any element $\lambda \in \mathbb{A}_{R,Z}^1$. If $\ker |\cdot|_\lambda \cap R \neq \{0_R\}$, the argument of Proposition 2.4 tells us that $\lambda \in \mathbb{A}_F^1$. Suppose that $\ker |\cdot|_\lambda \cap R = \{0_R\}$. As in the proof of Proposition 2.4, there is a unique positive number $\omega \in [1, \infty)$ such that λ extends to an element $\bar{\lambda} \in \mathbb{A}_{K^\omega}^1$. Since $\ker |\cdot|_{\bar{\lambda}}$ (which contains $\ker |\cdot|_\lambda$) is a non-zero prime ideal of $K[t]$ and K is algebraically closed, one can find a (unique) element $s \in K$ with $\ker |\cdot|_{\bar{\lambda}} = (\mathbf{t} - s) \cdot K[t]$ and it is not hard to check that $\lambda = \zeta_{s,0}^\omega$. \square

In the following, we will use Theorem 2.6 to obtain the Berkovich spectra of Banach group rings of finite cyclic groups over R . Let G be a finite abelian group and $(S, \|\cdot\|)$ be a commutative unital Banach ring. We denote by $S[G]$ the group ring of G over S , and endowed it with the norm $\|\sum_{g \in G} a_g g\| := \max_{g \in G} \|a_g\|$. Clearly, $S[G]$ is a commutative unital Banach ring.

Corollary 2.13. *Let R , F , Q and $(K, |\cdot|)$ be as in Theorem 2.6. Denote by $|\cdot|_0$ the trivial norm on F . Let G be a cyclic group of order M with u being a generator of G . Suppose b_1, \dots, b_n are all the distinct M -th roots of unity in K .*

(a) If we set $\left| \sum_{l=0}^{M-1} a_l u^l \right|_{\alpha_{b_k}^\omega} := \left| \sum_{l=0}^{M-1} a_l b_k^l \right|^\omega$ and $\left| \sum_{l=0}^{M-1} a_l u^l \right|_{\beta_{Q(b_k)}} := \left| \sum_{l=0}^{M-1} Q(a_l b_k^l) \right|_0$, then $\mathcal{M}(R[G]) = \{\alpha_{b_k}^\omega : \omega \in [1, \infty); k = 1, \dots, n\} \cup \{\beta_{Q(b_k)} : k = 1, \dots, n\}$.

(b) As a topological space, $\mathcal{M}(R[G])$ consists of n intervals of the form $[1, \infty]$ corresponding to the elements b_1, \dots, b_n such that the “1-ends” of all these intervals are free, while the “ ∞ -ends” of the two intervals corresponding to b_k and b_l are identified with each other if $Q(b_k) = Q(b_l)$.

Proof. (a) There is a contractive and surjective ring homomorphism

$$q_G : R\{\mathbf{t}\} \rightarrow R[G] \tag{15}$$

sending \mathbf{t} to u , and it is not hard to check that $\ker q_G = (\mathbf{t}^M - 1) \cdot R\{\mathbf{t}\}$. Hence, one may regard $\mathcal{M}(R[G])$ as a topological subspace of $\mathcal{M}(R\{\mathbf{t}\})$ through q_G in the following way:

$$\mathcal{M}(R[G]) = \{\lambda \in \mathbb{A}_R^1 : |\mathbf{t}|_\lambda \leq 1; |\mathbf{t}^M - 1|_\lambda = 0\} \subseteq \mathbb{A}_{R,Z}^1.$$

It is obvious that $\alpha_{b_k}^\omega$ and $\beta_{Q(b_k)}$ are well-defined elements in $\mathcal{M}(R[G])$, and they can be identified, respectively, with the elements $\zeta_{b_k,0}^\omega$ and $\zeta_{Q(b_k),0}$ in $\mathbb{A}_{R,Z}^1$.

On the other hand, let us pick an arbitrary element $\lambda \in \mathcal{M}(R[G])$. By Corollary 2.12(a), either $\lambda = \zeta_{s,0}^\omega$ for a unique $(s, \omega) \in K \times [1, \infty)$ or $\lambda \in \mathbb{A}_F^1$. In the first case, the condition $|\mathbf{t}^M - 1|_{\zeta_{s,0}^\omega} = 0$ will force $s = b_k$ for some $k \in \{1, \dots, n\}$, which means that $\lambda = \alpha_{b_k}^\omega$. In the second case, there exist $x \in F$ and $\tau \in \mathbb{R}_+$ satisfying $\lambda = \zeta_{x,\tau}$, and the condition $|\mathbf{t}^M - 1|_{\zeta_{x,\tau}} = 0$ tells us that $\tau = 0$ and $x^M = 1$. Since $Q(b_1), \dots, Q(b_n)$ are all the roots of $\mathbf{t}^M - 1$ in F , we conclude that $\lambda = \beta_{Q(b_k)}$ for some $k = 1, \dots, n$.

(b) By Corollary 2.12(a), it is not hard to see that the subset $\{\zeta_{b_k,0}^\omega : \omega \in [1, \infty); k = 1, \dots, n\}$ of $\mathbb{A}_{R,Z}^1$ are n disjoint intervals of the form $[1, \infty)$. Assume that $(s_i, \omega_i) \in \{b_1, \dots, b_n\} \times [1, \infty)$ ($i \in \mathcal{I}$) such that $\{\zeta_{s_i,0}^{\omega_i}\}_{i \in \mathcal{I}}$ converges to $\zeta_{Q(b_k),0}$ for some $k \in \{1, \dots, n\}$. Then Corollary 2.12(c) tells us that $|s_i - b_k| < 1$ eventually. In other words, $Q(s_i) = Q(b_k)$ eventually. Conversely, it follows from Corollary 2.12(b) that the conditions $Q(s_i) = Q(b_k)$ for large i and $\omega_i \rightarrow \infty$ will imply $\zeta_{s_i,0}^{\omega_i} \rightarrow \zeta_{Q(b_k),0}$. This completes the proof. \square

In the case when the field K is not algebraically closed, one may use the above as well as [3, Corollary 1.3.6] to describe $\mathcal{M}(R[G])$. In the following, we will consider the case when R is the ring \mathbb{Z}_p of p -adic integers and G is a cyclic p -group (for a fixed prime number p). Let us start with the following possibly known lemma.

Lemma 2.14. *For any $l \in \mathbb{Z}_+$, the polynomial $\mathbf{q}_{l+1} := \mathbf{t}^{p^l(p-1)} + \mathbf{t}^{p^l(p-2)} + \dots + \mathbf{t}^{p^l} + 1$ is irreducible in $\mathbb{Z}_p[\mathbf{t}]$.*

Proof. We first consider the case when $l = 0$. The equalities

$$((\mathbf{t} + 1) - 1) \cdot \sum_{i=1}^p (\mathbf{t} + 1)^{p-i} = (\mathbf{t} + 1)^p - 1 = \mathbf{t} \cdot \sum_{i=1}^p \binom{p}{i-1} \mathbf{t}^{p-i},$$

gives $\sum_{i=1}^p (\mathbf{t} + 1)^{p-i} = \sum_{i=1}^p \binom{p}{i-1} \mathbf{t}^{p-i}$. Thus, it follows from the Eisenstein's criterion that the polynomial $\sum_{i=1}^p (\mathbf{t} + 1)^{p-i}$ is irreducible in $\mathbb{Z}_p[\mathbf{t}]$, and hence so is $\sum_{i=1}^p \mathbf{t}^{p-i}$.

In the case of $l \geq 1$, we set $\mathbf{s} := (\mathbf{t} + 1)^{p^l} - 1$. Since $\sum_{i=1}^p (\mathbf{s} + 1)^{p-i} = \sum_{i=1}^p \binom{p}{i-1} \mathbf{s}^{p-i}$, we have

$$\sum_{i=1}^p (\mathbf{t} + 1)^{p^l(p-i)} = \sum_{i=1}^p \binom{p}{i-1} ((\mathbf{t} + 1)^{p^l} - 1)^{p-i}.$$

From the left hand side, we see that the coefficient of $\mathbf{t}^{p^l(p-1)}$ is 1 and the constant coefficient is p . From the right hand side, we know that all the other coefficients of \mathbf{t}^k are divisible by p . Thus, the Eisenstein's criterion tells us that $\sum_{i=1}^p (\mathbf{t} + 1)^{p^l(p-i)}$ is an irreducible polynomial in $\mathbb{Z}_p[\mathbf{t}]$ and hence so is $\sum_{i=1}^p \mathbf{t}^{p^l(p-i)}$. \square

Example 2.15. *Let G be the cyclic group of order p^N for a positive integer N . We set $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ and consider $q_G : \mathbb{Z}_p\{\mathbf{t}\} \rightarrow \mathbb{Z}_p[G]$ to be the canonical quotient map. As in the argument of Corollary 2.13, we may identify, through q_G :*

$$\mathcal{M}(\mathbb{Z}_p[G]) = \left\{ \lambda \in Q^\mathbb{A}(\mathbb{A}_{\mathbb{F}_p}^1) \cup \bigcup_{\omega \in [1, \infty)} J_\omega^\mathbb{A}(\mathbb{A}_{\mathbb{Q}_p}^1) : |\mathbf{t}|_\lambda \leq 1; |\mathbf{t}^{p^N} - 1|_\lambda = 0 \right\}$$

(see Proposition 2.4). By Lemma 2.14, the prime factorization of $\mathbf{t}^{p^N} - 1$ in $\mathbb{Z}_p[\mathbf{t}]$ is

$$\mathbf{t}^{p^N} - 1 = \mathbf{q}_0 \cdot \mathbf{q}_1 \cdot \mathbf{q}_2 \cdots \mathbf{q}_N,$$

where $\mathbf{q}_0 := \mathbf{t} - 1$. On the other hand, the following is the prime factorization of $\mathbf{t}^{p^N} - 1$ in $\mathbb{F}_p[\mathbf{t}]$:

$$\mathbf{t}^{p^N} - 1 = (\mathbf{t} - 1)^{p^N}. \quad (16)$$

Let \mathbb{C}_p be the completion of the algebraic closure of \mathbb{Q}_p , let R_p be the ring of integers of \mathbb{C}_p and let F_p be the residue field of R_p . For $1 \leq k \leq N$, we consider $\{r_{k,1}, \dots, r_{k,n_k}\}$ to be the set of all distinct roots of \mathbf{q}_k in \mathbb{C}_p . It follows from (16) that $\mathbf{t}^{p^N} - 1 = (\mathbf{t} - 1)^{p^N}$ in $F_p[\mathbf{t}]$. Hence, $Q(r_{k,i}) = 1$ for all possible k and i . Therefore, Corollary 2.13(a) tells us that

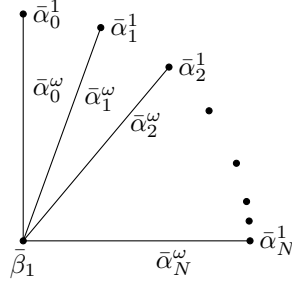
$$\mathcal{M}(R_p[G]) = \{\alpha_{r_{k,i}}^\omega : \omega \in [1, \infty); 1 \leq k \leq N; 1 \leq i \leq n_k\} \cup \{\alpha_1^\omega : \omega \in [1, \infty)\} \cup \{\beta_1\}.$$

As in Corollary 2.13(b), the topological space $\mathcal{M}(R_p[G])$ consists of $1 + \sum_{k=1}^N n_k$ intervals of the form $[1, \infty]$ with all the “1-ends” being free but with all the “ ∞ -ends” being identified with one point, namely, β_1 .

Again, the prime factorization as in (16) ensures that $\mathcal{M}(\mathbb{Z}_p[G]) \cap Q^\mathbb{A}(\mathbb{A}_{\mathbb{F}_p}^1) = \{Q^\mathbb{A}(\zeta_{1_{\mathbb{F}_p}, 0})\}$, and the semi-norm $\bar{\beta}_1 := Q^\mathbb{A}(\zeta_{1_{\mathbb{F}_p}, 0})$ coincides with the one induced by $\beta_1 \in \mathcal{M}(R_p[G])$ through restriction. On the other hand, as in [3, Corollary 1.3.6], any element $\lambda \in \mathcal{M}(\mathbb{Z}_p[G]) \cap J_\omega^\mathbb{A}(\mathbb{A}_{\mathbb{Q}_p^\omega}^1)$ can be extended to an element $\bar{\lambda} \in J_\omega^\mathbb{A}(\mathbb{A}_{\mathbb{C}_p^\omega}^1)$. It follows from $|\mathbf{t}|_{\bar{\lambda}} \leq 1$ and $|\mathbf{t}^{p^N} - 1|_{\bar{\lambda}} = 0$ that $\bar{\lambda}$ is either α_1^ω or $\alpha_{r_{k,i}}^\omega$ for suitable k and i .

Let us set $\bar{\alpha}_0^\omega$ to be the element in $\mathcal{M}(\mathbb{Z}_p[G])$ induced by α_1^ω . On the other hand, for a fixed $k \in \{1, 2, \dots, N\}$, by considering an automorphism in the Galois group of the splitting field of the irreducible polynomial \mathbf{q}_k over \mathbb{Q}_p , we know that $\alpha_{r_{k,i}}^\omega$ and $\alpha_{r_{k,j}}^\omega$ restrict to the same element in $\mathcal{M}(\mathbb{Z}_p[G])$ for any $i, j \in \{1, \dots, n_k\}$. We denote the resulting element by $\bar{\alpha}_k^\omega$. As $\bar{\alpha}_k^\omega$ comes from different irreducible factors of $\mathbf{t}^{p^N} - 1$ for different k , they are all distinct.

Consequently, as a quotient space of $\mathcal{M}(R_p[G])$, the topological space $\mathcal{M}(\mathbb{Z}_p[G])$ is the following subspace of \mathbb{R}^2 :



ACKNOWLEDGEMENT

This work is supported by the National Natural Science Foundation of China (11471168).

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